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Bond percolation processes in d dimensions

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Abstract. We study bond percolation processes on a d-dimensional simple hypercubic lattice. Exact expansions for the mean number of clusters, $K(p)$, and the mean cluster size, $S(p)$, in powers of $1/\sigma$, where $\sigma=2d-1$ and $p<p_c$, are derived through fifth and fourth order, respectively. The zeroth-order terms are the Bethe approximations. The critical probability p_c is found to have the expansion, probably asymptotic,

$$p_c=\sigma^{-1}(1+2\tfrac{1}{2}\sigma^{-2}+7\tfrac{1}{2}\sigma^{-3}+57\sigma^{-4}+\ldots),$$

while the cluster growth parameter λ can be expanded as

$$\lambda=\lambda_B(1-2\sigma^{-2}-\ldots)$$

where λ_B is the Bethe approximation for λ.

We also present series data for the mean cluster size and the cluster growth function for $d=4$ to 7. Numerical analysis suggests that the critical dimension, d_c, for bond percolation is $d_c=6$, as it seems to be for the site problem. The evidence also supports the conjecture that the value of a particular critical exponent in a given dimension is the same for both bond and site processes.

1. Introduction and summary

In this paper we continue our study of percolation processes on simple hypercubic lattices of coordination number $\nu=2d$, where d is the lattice dimensionality. In our earlier work (Gaunt *et al* 1976, to be referred to as GSR) we concentrated on site percolation and now we turn our attention to the corresponding bond problem.

The general techniques available for series development in higher d have been described by Fisher and Gaunt (1964) in their study of the Ising model and the excluded volume problem in d dimensions. The derivation of low density expansions for the bond percolation problem has been described by Sykes and Glen (1976). In combining these methods, we have followed closely the procedure described in detail by GSR for site percolation. For this reason, we simply focus on the results and compare them with the analogous results for the site problem. The reader who would like further details and comments should refer to GSR. For a general review of percolation processes, see Essam (1972).

We have divided the remainder of this paper into two sections. In § 2 we derive low density expansions through p^9 for the mean cluster size, $S(p)$, on hypercubic lattices of dimensionality $d=4$ to 7. These are analysed to determine $p_c(d)$ and $\gamma(d)$ defined by

$$S(p)\sim C(p_c-p)^{-\gamma},\qquad (p\to p_c-). \tag{1.1}$$

We have also calculated the total number, N_b, of connected clusters of b bonds through N_{10} for $d=4$ to 7. Assuming an asymptotic form of the usual kind

$$N_b \sim A b^{-\theta} \lambda^b, \qquad (b \to \infty) \tag{1.2}$$

we have estimated $\lambda(d)$ and $\theta(d)$.

As d increases, it appears not unreasonable that γ and θ attain their classical values of $\gamma = 1$ and $\theta = 2\frac{1}{2}$ at $d = 6$. Thus, our analysis for the bond problem supports the suggestion first made by Toulouse (1974) that the critical dimension, d_c, for percolation processes is $d_c = 6$. The same conclusion was reached by GSR for the site problem. Furthermore, our estimates of $\gamma(d)$ and $\theta(d)$ for the bond problem agree (to within the numerical uncertainties) with the corresponding estimates of GSR for the site problem. This supports the conjecture that the critical exponents for corresponding bond and site problems on the same lattice are identical.

We also give in (2.4) of § 2 the first three coefficients in a binomial expansion for $N_b(d)$ valid for general d and general $b \geq 3$. A similar expansion is possible for the coefficient $b'_r(d)$ of p^r in the low density expansion of $\binom{d}{1}pS(p)$; we give in (2.9) the first five coefficients for general $r \geq 8$. In § 3, we use these general binomial expansions to derive expansions in inverse powers of $\sigma = \nu - 1 = 2d - 1$ for the growth parameter λ and the critical probability p_c (see (3.7) and (3.15), respectively). As discussed by GSR these expansions are probably only asymptotic; nevertheless, they yield good approximations even when $d = 3$. We also derive $1/\sigma$ expansions valid for $p < p_c$ for the mean cluster size, $S(p)$, and the mean number of clusters, $K(p)$ (see (3.21) and (3.19), respectively). The $1/\sigma$ expansions are compared and contrasted with the corresponding expansions of GSR for the site problem and with the analogous expansions derived by Fisher and Gaunt (1964) for the Ising ferromagnet. As expected from the work of Kasteleyn and Fortuin (1969), we find that the Ising/percolation analogy is much closer for bond percolation that it is for site percolation.

2. Series expansions

For the general d-dimensional simple hypercubic lattice, we have derived the first eight perimeter polynomials D_n, one more than GSR for the corresponding site problem. They are, writing $q = 1 - p$,

$$D_1 = q^{4d-2}\binom{d}{1}$$

$$D_2 = q^{6d-4}\left[\binom{d}{1} + 4\binom{d}{2}\right]$$

$$D_3 = q^{8d-6}\left[\binom{d}{1} + (16 + 4q^{-1})\binom{d}{2} + 32\binom{d}{3}\right]$$

$$D_4 = q^{10d-8}\left[\binom{d}{1} + (53 + 32q^{-1} + q^{-2d})\binom{d}{2} + (324 + 96q^{-1})\binom{d}{3} + 400\binom{d}{4}\right]$$

$$\begin{aligned} D_5 = q^{12d-10}\Big[\binom{d}{1} &+ (172 + 160q^{-1} + 30q^{-2} + 8q^{-2d})\binom{d}{2} \\ &+ (2448 + 1512q^{-1} + 180q^{-2} + 24q^{-2d})\binom{d}{3} + (8064 + 2304q^{-1})\binom{d}{4} \\ &+ 6912\binom{d}{5}\Big] \end{aligned}$$

$$\begin{aligned} D_6 = q^{14d-12}\Big[\binom{d}{1} &+ (568 + 672q^{-1} + 332q^{-2} + 40q^{-2d} + 14q^{-2d-1})\binom{d}{2} \\ &+ (17041 + 15600q^{-1} + 4704q^{-2} + 400q^{-3} + 376q^{-2d} + 84q^{-2d-1})\binom{d}{3} \\ &+ (112824 + 63744q^{-1} + 9408q^{-2} + 576q^{-2d})\binom{d}{4} \\ &+ (239120 + 62720q^{-1})\binom{d}{5} + 153664\binom{d}{6}\Big] \end{aligned}$$

$$D_7 = q^{16d-14}[\tbinom{d}{1} + (1906 + 2712q^{-1} + 2030q^{-2} + 336q^{-3} + 168q^{-2d} + 156q^{-2d-1} + 2q^{-4d})\tbinom{d}{2} + (116004 + 137736q^{-1} + 67812q^{-2} + 15096q^{-3} + 384q^{-5} + 3864q^{-2d} + 2208q^{-2d-1} + 264q^{-2d-2} + 12q^{-4d})\tbinom{d}{3} + (1382400 + 1141248q^{-1} + 350400q^{-2} + 40256q^{-3} + 15840q^{-2d} + 4416q^{-2d-1})\tbinom{d}{4} + 5445120 + 2769920q^{-1} + 407040q^{-2} + 15680q^{-2d})\tbinom{d}{5} + (8257536 + 1966080q^{-1})\tbinom{d}{6} + 4194304\tbinom{d}{7}]$$

$$D_8 = q^{18d-16}[\tbinom{d}{1} + (6471 + 10880q^{-1} + 9972q^{-2} + 4064q^{-3} + 192q^{-4} + 677q^{-2d} + 958q^{-2d-1} + 228q^{-2d-2} + 22q^{-4d})\tbinom{d}{2} + (787965 + 1140576q^{-1} + 755532q^{-2} + 287280q^{-3} + 28704q^{-4} + 9216q^{-5} + 33996q^{-2d} + 31908q^{-2d-1} + 10080q^{-2d-2} + 408q^{-2d-4} + 312q^{-4d} + 72q^{-4d-1})\tbinom{d}{3} + (15998985 + 17116800q^{-1} + 7855008q^{-2} + 1932864q^{-3} + 114816q^{-4} + 24576q^{-5} + 282216q^{-2d} + 164928q^{-2d-1} + 26880q^{-2d-2} + 624q^{-4d})\tbinom{d}{4} + (104454120 + 77177280q^{-1} + 22232640q^{-2} + 2609280q^{-3} + 688640q^{-2d} + 192000q^{-2d-1})\tbinom{d}{5} + (280717488 + 128770560q^{-1} + 17729280q^{-2} + 491520q^{-2d})\tbinom{d}{6} + (326265408 + 70543872q^{-1})\tbinom{d}{7} + 136048896\tbinom{d}{8}]. \tag{2.1}$$

These polynomials have been derived by the yield factor technique described by Sykes *et al* (1976b) using the general configurational data for the site problem through nine sites derived independently by M F Sykes and collaborators (private communication). For the square ($d = 2$) and simple cubic ($d = 3$) lattice, these expressions reduce correctly to the known results which extend to D_{13} and D_9, respectively (unpublished work by Sykes and Glen 1976, Sykes *et al* 1976c). The form of these polynomials differs from the corresponding form found by GSR for the site problem by the presence of $(1/q)$-factors raised to d-dependent powers. Essentially, these factors occur because clusters with a given number of bonds can have various numbers of sites. Thus the first of these factors, namely q^{-2d} in D_4, arises from the square of 4 bonds (and 4 sites) of which there are $\tbinom{d}{2}$ on the lattice, all the remaining contributions to D_4 coming from clusters of 4 bonds and 5 sites.

Using the general method outlined by Essam and Sykes (1966) we have also derived an expansion for the mean number of clusters in general form through p^{11}, and find

$$K(p) = \tbinom{d}{1}p + [-\tbinom{d}{1} - 4\tbinom{d}{2}]p^2 + [4\tbinom{d}{2} + 8\tbinom{d}{3}]p^3 + [-12\tbinom{d}{3} - 16\tbinom{d}{4}]p^4 + [6\tbinom{d}{3} + 32\tbinom{d}{4} + 32\tbinom{d}{5}]p^5 + [2\tbinom{d}{2} + 15\tbinom{d}{3} - 24\tbinom{d}{4} - 80\tbinom{d}{5} - 64\tbinom{d}{6}]p^6 + [-2\tbinom{d}{2} - 12\tbinom{d}{3} + 8\tbinom{d}{4} + 80\tbinom{d}{5} + 192\tbinom{d}{6} + 128\tbinom{d}{7}]p^7 + [7\tbinom{d}{2} + 162\tbinom{d}{3} + 647\tbinom{d}{4} - 40\tbinom{d}{5} - 240\tbinom{d}{6} - 448\tbinom{d}{7} - 256\tbinom{d}{8}]p^8 + [-12\tbinom{d}{2} - 292\tbinom{d}{3} - 800\tbinom{d}{4} + 10\tbinom{d}{5} + 160\tbinom{d}{6} + 672\tbinom{d}{7} + 1024\tbinom{d}{8} + 512\tbinom{d}{9}]p^9 + [28\tbinom{d}{2} + 1950\tbinom{d}{3}$$

$$+20576\binom{d}{4}+47615\binom{d}{5}-60\binom{d}{6}-560\binom{d}{7}-1792\binom{d}{8}-2304\binom{d}{9}$$
$$-1024\binom{d}{10}]p^{10}+[-54\binom{d}{2}-4980\binom{d}{3}-45504\binom{d}{4}-71040\binom{d}{5}$$
$$+12\binom{d}{6}+280\binom{d}{7}+1792\binom{d}{8}+4608\binom{d}{9}+5120\binom{d}{10}$$
$$+2048\binom{d}{11}]p^{11}+\ldots \tag{2.2}$$

The corresponding expansion for site percolation processes is given by GSR through order one less.

The procedures described by Sykes and Glen (1976) enable us, by using (2.1) and the first ten coefficients of (2.2), to obtain the total number of clusters with b bonds, N_b, through N_{10} for all d. Explicit values are given in table 1 for $d=2$ to 7. The additional data for $d=2$ and 3 was obtained using the extra perimeter polynomials and extra coefficients in the $K(p)$ expansion available in these low dimensions (unpublished work by Sykes and Glen 1976, Sykes *et al* 1976c).

Alternatively, the data of table 1 can be summarised for all dimensions in the form

$$N_b(d)=\sum_{\xi=0}^{b-1}\alpha_\xi^b\binom{d}{b-\xi} \tag{2.3}$$

$$=2^b(b+1)^{b-2}\binom{d}{b}+2^{b-2}(b+1)^{b-4}(b-1)(b+1)(2b-1)\binom{d}{b-1}$$
$$+\left(2^{b-4}(b+1)^{b-6}(b-2)(b+1)\frac{(12b^4-20b^3-33b^2-46b+195)}{6}\right.$$
$$\left.+2^{b-3}b^{b-5}(b-2)(b-3)\right)\binom{d}{b-2}+\ldots+\binom{d}{1}, \qquad (b\geqslant 3). \tag{2.4}$$

The calculation of successive α_ξ^b numerically for values of b through $b=10$ is simply a matter of arithmetic. In general, the derivation of the α_ξ^b as functions of b is more difficult than the calculation of the A_ξ^s for the corresponding site problem, for which GSR also give the first three terms. However, contributions to α_0^b, like those to $A_1^s=2^{s-1}s^{s-3}$, come from Cayley trees, since these are the only clusters which can enter all dimensions. Now Cayley trees with b bonds have $(b+1)$ sites, and hence it follows immediately that $\alpha_0^b=A_1^{b+1}$. The second and third terms of (2.4) are confirmed by the data of table 1. We saw earlier in connection with (2.1) that when the square cluster first appears it gives rise to a new feature in the bond problem, namely, a d-dependent power of $(1/q)$. Likewise, the second (and smaller) contribution to α_2^b arises from clusters consisting of a single square of 4 bonds and a tail of $(b-4)$ bonds, each bond of which extends into a new dimension, resulting in a 'fully-stretched' cluster which is only embeddable in a $(b-4)+2(=b-2)$-dimensional space. The fact that some of the α_ξ^b can only be written as the sum of different functions of b is another feature of the bond problem not shared with the A_ξ^s of the site problem. Notice, however, that the dominant contribution to α_ξ^b has the same structure as $A_{\xi+1}^{b+1}$, namely

$$2^{b-2\xi}(b+1)^{b-2(\xi+1)}(b-\xi)P_{3\xi-1}(b), \qquad (\xi=0,1,2,\ldots) \tag{2.5}$$

where $P_m(b)$ is a polynomial in b of degree m and $P_{-1}(b)$ is interpreted as b^{-1} in α_0^b and A_1^{b+1}. The form of P_2 and P_5 in (2.4) suggests that for the bond problem

$$P_{3\xi-1}(b)=(b+1)p_{3\xi-2}(b) \tag{2.6}$$

where $p_{3\xi-2}(b)$ is another polynomial of degree one less than $P_{3\xi-1}(b)$. Clearly, the last term in (2.4) corresponds to the solution of the linear chain and is $\alpha_{b-1}^b=1$.

Table 1. Total number, N_b, of bond clusters with b bonds per site of a d-dimensional simple hypercubic lattice.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
N_1	2	3	4	5	6	7
N_2	6	15	28	45	66	91
N_3	22	95	252	525	946	1 547
N_4	88	681	2 600	7 065	15 696	30 513
N_5	372	5 277	29 248	104 097	285 828	661 549
N_6	1 628	43 086	349 132	1 632 915	5 551 480	15 314 936
N_7	7 312	365 313	4 351 944	26 817 465	113 045 832	372 033 993
N_8	33 466	3 186 444	56 062 681	456 137 580	2 386 821 009	9 377 038 237
N_9	155 446	28 414 802	741 132 648	7 975 932 715	51 856 494 126	243 337 296 804
N_{10}	730 534	257 908 020	10 003 089 696	142 619 162 000	1 153 039 934 712	6 465 642 398 915
N_{11}	3 466 170	2 375 037 477				
N_{12}	16 576 874					
N_{13}	79 810 756					
N_{14}	386 458 826					
N_{15}	1 880 580 352					

Table 3. Estimates of critical parameters for bond percolation processes.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$p_c(d)$	0·5	0·247 ± 0·003	0·161 ± 0·0015	0·118 ± 0·001	0·0941 ± 0·0005	0·0786 ± 0·0002
p_c (Fisch)	0·5	0·2465 ± 0·0002	0·1600 ± 0·0002	0·1181 ± 0·0002	0·0943 ± 0·0002	0·0788 ± 0·0002
$p_c^{(\sigma)}(d)$	0·5556	0·2488	0·1576	0·1180	0·09415	0·07869
$\gamma(d)$	2·425 ± 0·005	1·66 ± 0·07	1·48 ± 0·08	1·18 ± 0·07	1·04 ± 0·06	1·00 ± 0·03
$\lambda(d)$	5·210 ± 0·006	10·62 ± 0·08	16·3 ± 0·4	22·1 ± 0·8	27·75 ± 1.0	33·25 ± 1·5
$\lambda^{(\sigma)}(d)$	5·250	11·230	16·931	22·522	28·060	33·567
$\theta(d)$	1·00 ± 0·01	1·55 ± 0·05	1·90 ± 0·07	2·2 ± 0·1	2·3 ± 0·2	2·4 ± 0·2

We have also obtained using (2.1) and (2.2) the expansion for the mean size of finite clusters, $S(p)$, given by

$$\binom{d}{1} pS(p) = \sum_{r=1}^{\infty} b'_r p^r \tag{2.7}$$

for all d through b'_{10}. Explicit values are given in table 2 for $d = 2$ to 7. The supplementary data for $d = 2$ and 3 is taken from Sykes and Glen (1976) and Sykes *et al* (1976c), respectively. Note that it is important to work with the coefficients b'_r rather than the coefficients b_r of $S(p)$ itself as done by GSR for the site problem. This difference in treatment arises because in the bond problem physical quantities are defined 'per lattice bond' rather than 'per lattice site' as in the site problem.

Table 2. Coefficients b'_r in the expansion of $\binom{d}{1} pS(p) = \sum_{r \geq 1} b'_r p^r$ for a d-dimensional simple hypercubic lattice.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
b'_1	2	3	4	5	6	7
b'_2	12	30	56	90	132	182
b'_4	36	150	392	810	1 452	2 366
b'_4	96	714	2 672	7 170	15 792	30 506
b'_5	252	3 342	18 008	62 970	170 772	391 622
b'_6	600	14 994	119 168	547 890	1 836 792	5 010 530
b'_7	1 524	67 686	788 456	4 762 890	19 740 252	64 065 134
b'_8	3 336	294 522	5 141 120	41 133 570	211 444 392	817 617 626
b'_9	8 432	1 304 682	33 636 752	355 513 730	2 265 196 512	10 434 586 442
b'_{10}	17 336	5 566 038	217 379 600	3 057 241 370	24 210 714 792	133 011 279 086
b'_{11}	43 976	24 376 170				
b'_{12}	86 116					
b'_{13}	221 664					
b'_{14}	404 864					
b'_{15}	1 122 040					

The data of table 2 can be summarised for all d in the alternative form

$$b'_r(d) = \sum_{\xi=0}^{r-1} \beta'^{r}_{\xi} \binom{d}{r-\xi} \tag{2.8}$$

$$\begin{aligned} &= 2^r r! \binom{d}{r} + 2^{r-2}(r-1)!\,2(r-1)^2 \binom{d}{r-1} \\ &\quad + 2^{r-4}(r-2)!\,\tfrac{1}{3}(6r^4 - 40r^3 + 96r^2 - 128r + 138)\binom{d}{r-2} \\ &\quad + 2^{r-6}(r-3)!\,\tfrac{1}{3}(4r^6 - 56r^5 + 316r^4 - 976r^3 + 1996r^2 - 3084r + 3360)\binom{d}{r-3} \\ &\quad + 2^{r-8}(r-4)!\,\tfrac{2}{45}(15r^8 - 360r^7 + 3680r^6 - 21312r^5 + 79610r^4 \\ &\quad - 212100r^3 + 440180r^2 - 713433r + 760860)\binom{d}{r-4} \\ &\quad + \ldots + 2\binom{d}{1}, \qquad (r \geq 8). \end{aligned} \tag{2.9}$$

The numerical calculation of successive β'^{r}_{ξ} through $r = 10$ is again a matter of arithmetic. The derivation of the general form (2.9) is not as difficult as the derivation of (2.4) since it appears that none of the β_ξ^r has to be written as the sum of different functions of r. We have derived the first five $\beta_\xi^r(\xi = 0, 1, 2, 3, 4)$, while for the corresponding expansion for the site problem, GSR give the first four $B_\xi^r(\xi = 0, 1, 2, 3)$. Note that $\beta_0^r = B_0^r$, and that β_ξ^r and B_ξ^r have the same general structure

$$2^{r-2\xi}(r-\xi)!\,Q_{2\xi}(r), \qquad (\xi = 0, 1, 2, \dots) \tag{2.10}$$

where $Q_{2\xi}(r)$ is a polynomial in r of degree 2ξ. Clearly the last term in (2.9) corresponds to the solution of the linear chain and is $\beta_{r-1}^r = 2$.

To end this section we report our analysis of the low density expansions of the mean cluster size, $S(p)$, and the expansions of the generating function, $A(z)$, for the total number of clusters with b bonds. The procedure that we have followed is based upon the ratio and Padé approximant techniques (Gaunt and Guttmann 1974) and is described in § 4 of GSR.

Our best overall estimates of p_c, λ, γ and θ for $d = 4$ to 7 are presented in table 3. For completeness the corresponding values for $d = 2$ and 3 are also tabulated. For the percolation problem, $p_c(2) = \frac{1}{2}$ is an exact result (Sykes and Essam 1963), $\gamma(2)$ is taken from Sykes *et al* (1976a), while $p_c(3)$ and $\gamma(3)$ are from Sykes *et al* (1976c). Independent estimates of p_c(d) due to Fisch (1977) are also tabulated; the agreement with our estimates is seen to be excellent. For the total number of clusters, $\lambda(2)$ is from Sykes and Glen (1976) although we have increased their uncertainty by 50% because of the uncertainty of 0·01 that we have estimated for their estimate of $\theta(2)$. The value $\lambda(3) = 10{\cdot}62$ agrees with Sykes *et al* (1976c), although our uncertainty is larger because of our estimate of the uncertainty in $\theta(3)$. Sykes *et al* (1976c) simply found that 'θ is about $\frac{3}{2}$'. Our central estimate of θ, which we obtained by applying the same methods that we have used for $d \geq 4$, is in precise agreement with a recent estimate of Guttmann and Gaunt (1978) obtained in a completely different way.

The results of table 3 confirm that to within numerical accuracy the critical exponents γ and θ attain their classical values of $\gamma = 1$ and $\theta = 2\frac{1}{2}$ in six dimensions, that is, $d_c = 6$. The same conclusion was drawn by GSR for site percolation processes. Furthermore, the estimates of $\gamma(d)$ and $\theta(d)$ for $d = 2$ to 6 agree to within the numerical uncertainties with those in table 3 of GSR for the corresponding site problem. This provides further support for the universality of critical exponents in percolation theory.

3. Expansions in $1/\sigma$

By following closely the procedures outlined by GSR, we now derive expansions in the variable $1/\sigma$. First we use the $1/\sigma$ expansion for the binomial coefficient $\binom{d}{s}$ given in (3.1) of GSR and substitute into (2.4) giving

$$N_b(d) = \frac{(b+1)^{b-2}}{b!}\sigma^b\left[1 - \frac{b(b-5)}{2(b+1)}\sigma^{-1} + \frac{b(b-1)(b-2)}{24(b+1)^4}\right.$$
$$\left.\times\left(3b^3 - 79b^2 + 73b + 155 + \frac{12b^{b-5}(b-3)}{(b+1)^{b-6}}\right)\sigma^{-2} + \dots\right], \qquad (b \geq 3). \tag{3.1}$$

Formally taking the logarithm of the expression and using Stirling's formula for the factorial we find

$$\ln N_b(d) = b \ln \sigma + b - \tfrac{5}{2} \ln b - (\tfrac{1}{2} \ln 2\pi - 1) - 2\tfrac{7}{12} b^{-1} + O(b^{-2})$$
$$- \frac{b(b-5)}{2(b+1)} \sigma^{-1} - \left(\frac{b(32b^4 - 149b^3 + 171b^2 + 197b - 155)}{12(b+1)^4} - \frac{b^{b-4}(b-1)(b-2)(b-3)}{2(b+1)^{b-2}} \right) \sigma^{-2} - \ldots . \tag{3.2}$$

Hence, from the definition (1.2) it follows that

$$\ln \lambda(d) = \lim_{b\to\infty} \frac{1}{b} \ln N_b(d) = \ln \sigma + 1 - \tfrac{1}{2}\sigma^{-1} - 2\tfrac{1}{6}\sigma^{-2} - \ldots . \tag{3.3}$$

This expansion will continue in this way provided, as seems probable, that the higher coefficients of (3.2) are also of O(b) for b large. Using the rigorous results of Fisher and Essam (1961) it can be shown that in the Bethe approximation the total number of clusters per site with b bonds is given asymptotically by

$$N_b \sim B b^{-5/2} [\sigma^\sigma / (\sigma - 1)^{\sigma - 1}]^b, \qquad (b \to \infty) \tag{3.4}$$

where the amplitude

$$B = (\sigma/2\pi)^{1/2} (\sigma + 1) \sigma^\sigma (\sigma - 1)^{-(\sigma + \frac{3}{2})}. \tag{3.5}$$

Thus, in the Bethe approximation the growth parameter λ is the same for bond and site clusters, namely

$$\lambda_B = \sigma^\sigma / (\sigma - 1)^{\sigma - 1}. \tag{3.6}$$

From (3.3) and (3.6) we find

$$\lambda(\text{bond}) = \lambda_B (1 - 2\sigma^{-2} - \ldots). \tag{3.7}$$

The corresponding expansion for site percolation is given by (3.11) of GSR, namely

$$\lambda(\text{site}) = \lambda_B (1 - 1\tfrac{1}{2}\sigma^{-1} - 2\sigma^{-2} - \ldots). \tag{3.8}$$

Hence,

$$\frac{\lambda(\text{bond}) - \lambda(\text{site})}{\lambda_B} = 1\tfrac{1}{2}\sigma^{-1} + O(\sigma^{-3}) \tag{3.9}$$

suggesting that $\lambda(\text{bond}) > \lambda(\text{site})$, which is verified for $d = 2$ to 6 by the numerical results in table 3 and table 3 of GSR. The above inequality has recently been studied rigorously by Whittington and Gaunt (1978).

The most significant difference between (3.7) and (3.8) is that for the bond problem, as far the $1/\sigma$ expansions for the Ising critical point and self-avoiding walk limit (Fisher and Gaunt 1964), the leading correction to the Bethe approximation is of *second* order in $1/\sigma$, while for the site problem it is of first order. This is not entirely unexpected since we suspect from the work of Kasteleyn and Fortuin (1969) that bond percolation should provide a closer analogy with the Ising model and excluded volume problem than does site percolation.

Assuming (3.7) is asymptotic (as seems likely), then truncation after the smallest term for given σ should yield the optimum approximation. As occurred for the site problem, the expansion is so short that it is impossible to tell if the smallest term has been attained in any dimension. Consequently we have estimated λ by truncation after the last term; GSR found that such a procedure worked well for the site problem. The values obtained, $\lambda^{(\sigma)}$, are compared in table 3 with the best series estimates obtained in § 2. Surprisingly, the $1/\sigma$ expansion is very accurate even for $d=2$, $\lambda^{(\sigma)}$ being only 0·8% larger than the series estimate. The difference between λ and $\lambda^{(\sigma)}$ rises to a maximum of 5·7% for $d=3$ and then decreases monotonically to less than 1% again for $d=7$. For $d=5$, 6 and 7, values of $\lambda^{(\sigma)}$ fall within the numerical uncertainties of the series estimates. It is interesting that for bond percolation $\lambda^{(\sigma)}$ overestimates λ for all $d=2$ to 7, whereas for site percolation the opposite occurs.

A similar procedure can be followed for the coefficients $b_r'(d)$. Since we have one more term of (2.9) than of the corresponding expansion for site percolation, we first require one more term in the $1/\sigma$ expansion (3.1) of GSR for $\binom{d}{s}$, namely

$$+\tfrac{1}{360}s(s-1)(s-2)(s-3)(15s^4-150s^3+455s^2-468s+127)\sigma^{-4}+\ldots. \tag{3.10}$$

Substituting into (2.9) we find

$$b_r'(d)=\sigma^r[1+\sigma^{-1}+(-2\tfrac{1}{2}r+8\tfrac{1}{2})\sigma^{-2}+(-10r+57)\sigma^{-3}+(3\tfrac{1}{8}r^2-82\tfrac{5}{8}r+550)\sigma^{-4}+\ldots], \quad (r\geqslant 8). \tag{3.11}$$

Denoting the integer part of x by $[x]$, we see that the coefficient of σ^{-m} appears to be a polynomial in r of degree $[\frac{1}{2}m]$, as compared to degree m for site percolation. Taking the logarithm of (3.11) yields

$$\ln b_r'(d)=r\ln\sigma+\sigma^{-1}+(-2\tfrac{1}{2}r+8)\sigma^{-2}+(-7\tfrac{1}{2}r+48\tfrac{5}{6})\sigma^{-3}+(-53\tfrac{7}{8}r+465\tfrac{1}{8})\sigma^{-4}+\ldots, \quad (r\geqslant 8) \tag{3.12}$$

where the higher-order terms in r^2 have all cancelled. Assuming, as seems likely, that this cancellation will continue in the general term, the logarithm will be formally linear in r to all orders. This simple situation also occurs in the corresponding expansions for site percolation, the excluded volume problem and the Ising problem (GSR, Fisher and Gaunt 1964). For the total number of bond or site clusters, a more complicated situation obtains, as given by (3.2) and by (3.4) of GSR, in which the general term is only asymptotically linear.

Formally defining a limit μ by

$$\ln\mu(d)=\lim_{r\to\infty}\frac{1}{r}\ln b_r'(d) \tag{3.13}$$

we find

$$\ln\mu(d)=\ln\sigma-2\tfrac{1}{2}\sigma^{-2}-7\tfrac{1}{2}\sigma^{-3}-53\tfrac{7}{8}\sigma^{-4}-\ldots. \tag{3.14}$$

As for the site problem, the radius of convergence of the $S(p)$ series is determined by a singularity on the negative real p axis, at least for small enough d. However, following the arguments of GSR, we write $\mu=1/p_c$ and find from (3.14) that

$$p_c(\text{bond})=\sigma^{-1}(1+2\tfrac{1}{2}\sigma^{-2}+7\tfrac{1}{2}\sigma^{-3}+57\sigma^{-4}+\ldots). \tag{3.15}$$

The corresponding expansion for site percolation is given by GSR as

$$p_c(\text{site})=\sigma^{-1}(1+1\tfrac{1}{2}\sigma^{-1}+3\tfrac{3}{4}\sigma^{-2}+20\tfrac{3}{4}\sigma^{-3}+\ldots). \tag{3.16}$$

In both cases, the zeroth-order term is $1/\sigma$ which is the Bethe approximation for p_c for both bond and site percolation. As discussed earlier for the cluster growth parameter λ, the leading correction term is, as expected, of second order for bond percolation but first order for site percolation.

It follows from (3.15) and (3.16) that

$$p_c(\text{site}) - p_c(\text{bond}) = 1\tfrac{1}{2}\sigma^{-2} + 1\tfrac{1}{4}\sigma^{-3} + 13\tfrac{1}{4}\sigma^{-4} + \ldots \qquad (3.17)$$

suggesting $p_c(\text{site}) > p_c(\text{bond})$, which is verified for $d = 2$ to 6 by the numerical results in table 3 and table 3 of GSR. The above inequality is proved for the triangular lattice since this is the only case for which p_c is known exactly for both bond and site problems (Sykes and Essam 1963). The weaker result, $p_c(\text{site}) \geqslant p_c(\text{bond})$, has been proved rigorously by Hammersley (1961) for all lattices.

It is easy to demonstrate explicitly that the magnitude of the terms in (3.15) pass through a minimum for $d = 2$, 3 and 4. For $d = 5$ it is likely that the last term in (3.15) is the smallest. Assuming this to be the case and hence truncating after the last term yields an estimate which is too small by an amount equal to 1·4 times the smallest term. We have therefore calculated approximations, $p_c^{(\sigma)}$, to p_c from (3.15) by truncating after the smallest term and adding 1·4 times the smallest term *for all d*. For the purpose of these calculations we have assumed that the last term is the smallest for both $d = 6$ and 7; for $d = 6$ this assumption is possibly correct while for $d = 7$ it is probably wrong but should not affect the final result too much. The estimates obtained in this way are presented in table 3 where they are compared with the best series estimates obtained in § 2. The accuracy of $p_c^{(\sigma)}$ is worst (11% too large) for $d = 2$ as might be expected. For $d = 3$, $p_c^{(\sigma)}$ is around $\frac{3}{4}$% too large and for $d = 4$ is about 2% too small. For $d = 5$, $p_c^{(\sigma)}$ coincides with the series estimate by construction, while for $d = 6$ and 7, $p_c^{(\sigma)}$ lies well within the numerical uncertainties of the series estimates.

We have also derived $1/\sigma$ expansions for the mean number of clusters, $K(p)$, and for the mean cluster size, $S(p)$. For the mean number of clusters we must first subtract from (2.2) the corresponding expansion for the Bethe approximation, namely

$$\begin{aligned} K_B(p) = {} & \tbinom{d}{1}p - [\tbinom{d}{1} + 4\tbinom{d}{2}]p^2 + [4\tbinom{d}{2} + 8\tbinom{d}{3}]p^3 - [\tbinom{d}{2} + 12\tbinom{d}{3} + 16\tbinom{d}{4}]p^4 \\ & + [6\tbinom{d}{3} + 32\tbinom{d}{4} + 32\tbinom{d}{5}]p^5 - [\tbinom{d}{3} + 24\tbinom{d}{4} + 80\tbinom{d}{5} + 64\tbinom{d}{6}]p^6 \\ & + [8\tbinom{d}{4} + 80\tbinom{d}{5} + 192\tbinom{d}{6} + 128\tbinom{d}{7}]p^7 \\ & - [\tbinom{d}{4} + 40\tbinom{d}{5} + 240\tbinom{d}{6} + 448\tbinom{d}{7} + 256\tbinom{d}{8}]p^8 \\ & + [10\tbinom{d}{5} + 160\tbinom{d}{6} + 672\tbinom{d}{7} + 1024\tbinom{d}{8} + 512\tbinom{d}{9}]p^9 \\ & - [\tbinom{d}{5} + 60\tbinom{d}{6} + 560\tbinom{d}{7} + 1792\tbinom{d}{8} + 2304\tbinom{d}{9} + 1024\tbinom{d}{10}]p^{10} \\ & + [12\tbinom{d}{6} + 280\tbinom{d}{7} + 1792\tbinom{d}{8} + 4608\tbinom{d}{9} + 5120\tbinom{d}{10} + 2048\tbinom{d}{11}]p^{11} \\ & - \ldots . \end{aligned} \qquad (3.18)$$

Then by following the procedure detailed by GSR we find

$$\begin{aligned} K = K_B & + (\tfrac{1}{8}x^4)\sigma^{-2} + (\tfrac{1}{3}x^6)\sigma^{-3} + (-\tfrac{1}{8}x^4 - \tfrac{3}{4}x^6 - \tfrac{1}{4}x^7 + 1\tfrac{11}{16}x^8)\sigma^{-4} \\ & + (-\tfrac{1}{3}x^6 + \tfrac{1}{2}x^7 - 10\tfrac{1}{8}x^8 - 2\tfrac{1}{12}x^9 + 12\tfrac{2}{5}x^{10})\sigma^{-5} \\ & + (\tfrac{3}{4}x^6 + \tfrac{1}{4}x^7 + 14\tfrac{3}{8}x^8 + 10\tfrac{7}{12}x^9 - 132\tfrac{5}{12}x^{10} - 18\tfrac{1}{2}x^{11} + \ldots)\sigma^{-6} \\ & + (-\tfrac{1}{2}x^7 + 10\tfrac{1}{8}x^8 - 12\tfrac{5}{12}x^9 + 479\tfrac{23}{24}x^{10} + 159x^{11} + \ldots)\sigma^{-7} \\ & + (-16\tfrac{1}{16}x^8 - 10\tfrac{7}{12}x^9 - 484\tfrac{5}{24}x^{10} - 450\tfrac{3}{4}x^{11} + \ldots)\sigma^{-8} \end{aligned}$$

$$+(14\tfrac{1}{2}x^9-492\tfrac{43}{120}x^{10}+310\tfrac{1}{2}x^{11}+\ldots)\sigma^{-9}+(616\tfrac{5}{8}x^{10}+469\tfrac{1}{4}x^{11}+\ldots)\sigma^{-10}$$
$$+(-469\tfrac{1}{2}x^{11}+\ldots)\sigma^{-11}+\ldots, \qquad (x=\sigma p), \tag{3.19}$$

which is correct to order x^{11} and to order $(1/\sigma)^5$. Evidently the coefficients have the same form as they do for the site problem, namely, the coefficient of σ^{-m} is a polynomial in x, the term of lowest degree being x^m and that of highest degree being x^{2m}.

In as far as the truncated series in $1/\sigma$ is a good representation of $K(p)$ we may conclude that the Bethe approximation becomes more accurate as $\sigma\to\infty$. The leading correction term is again of second order in $1/\sigma$. The analogous expansion for the free energy $f=N^{-1}\ln Z$ of the Ising model (Fisher and Gaunt 1964) is

$$f=f_{\mathrm{B}}+(\tfrac{1}{8}x^4)\sigma^{-2}+(\tfrac{1}{3}x^6)\sigma^{-3}+(-\tfrac{1}{8}x^4-\tfrac{3}{4}x^6+1\tfrac{11}{16}x^8)\sigma^{-4}$$
$$+(-\tfrac{1}{3}x^6-9\tfrac{7}{8}x^8+12\tfrac{2}{5}x^{10})\sigma^{-5}+\ldots \tag{3.20}$$

where f_{B} is the Bethe approximation for $T>T_{\mathrm{c}}$. The correspondence between (3.19) and (3.20) is (as expected) even closer than that observed between (3.20) and the expansion corresponding to (3.19) for the site problem, namely (3.24) in GSR. The first two correction terms are identical while the correction terms of order $(1/\sigma)^4$ differ only by the term $-\frac{1}{4}x^7$ in (3.19). Even the terms of order $(1/\sigma)^5$ are not too different, the first and last coefficients being identical. As pointed out by GSR this similarity in form reflects the close formal analogy which exists between the free energy of the Ising ferromagnet and the mean number of clusters in the percolation problem (Kasteleyn and Fortuin 1969).

For the mean cluster size, $S(p)$, we find, following the methods outlined by GSR, that

$$S=S_{\mathrm{B}}-\frac{3x^3+2x^4}{(1-x)^2}\sigma^{-2}+\frac{3x^3+8x^4-18x^5-8x^6}{(1-x)^2}\sigma^{-3}$$
$$+\frac{-6x^4+66x^5+42x^6-261x^7+118x^8+53\tfrac{1}{2}x^9}{(1-x)^3}\sigma^{-4}+\ldots. \tag{3.21}$$

The leading term, S_{B}, is the Bethe approximation for $p<p_{\mathrm{c}}$,

$$S_{\mathrm{B}}(p)=(1+\sigma p)/(1-\sigma p), \qquad (p<p_{\mathrm{c}}) \tag{3.22}$$

which exhibits a simple pole at the Bethe critical point $x=1$ as is to be expected.

The analogous expansion for the zero-field reduced susceptibility of the Ising model (Fisher and Gaunt 1964) is

$$\chi=\chi_{\mathrm{B}}-\frac{x^4}{(1-x)^2}\sigma^{-2}+\frac{x^4-4x^6}{(1-x)^2}\sigma^{-3}+\ldots \tag{3.23}$$

where χ_{B} is the Bethe approximation for $T>T_{\mathrm{c}}$. Although the Ising/percolation analogy is not now so close as it was for the free energy/mean number, it is much closer for the bond problem than it is for the site problem (see (3.26) of GSR). The leading correction term in both (3.21) and (3.23) is second order in $(1/\sigma)$, but for the site problem it is first order. Furthermore, Fisher and Gaunt (1964) found that the coefficient of $(1/\sigma)^m$ in (3.23) diverges at $x=1$ like $(1-x)^{-1-[m/2]}$ as it does in (3.21), whereas for the site problem the divergence is like $(1-x)^{-(m+1)}$.

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